Explicit presentations for moduli stacks of n-smooth group schemes

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Let $\mathrm{Sm}_n^1/\mathbb{F}_p$ be the moduli stack of n-smooth group schemes of rank 1 (i.e. order p^n). Recall [KM, Theorem B] that these are smooth stacks over $\mathbb{F}_p$ and that the truncation morphisms $\mathrm{Sm}_n^1 \to \mathrm{Sm}_{n-1}^1$ are smooth. The goal of this note is to explain an explicit presentation for these stacks and truncation morphisms (Proposition 2.2).

1 n-cosmooth modules of rank 1

By [KM, Theorem A], the stacks Sm_n^1 may be identified with stacks of n-cosmooth modules of rank 1, and we systematically work with such modules in this note. For more background concerning n-smooth group schemes and n-cosmooth Dieudonné modules, we refer the reader to [KM]. In the sequel, R denotes an $\mathbb{F}_p$ algebra, $\mathbb{E}_R$ is the Cartier ring over R, and $\mathbb{E}_{n,R} := \mathbb{E}_R/V^n\mathbb{E}_R$ is the n-truncated Cartier ring over R.

1.1 Preliminary lemmas

The following lemma was key to the proof of smoothness of $\mathrm{Sm}_n^1 \to \mathrm{Sm}_{n-1}^1$ in [KM].

Lemma 1.1 ([KM], Lemma 6.0.2)**.** Let M be an n-cosmooth $\mathbb{E}_R$-module of rank 1 and $e \in M$ such that the image of e is a basis over R for M/VM. Then for all $m \in M$ there exist unique $a_i \in R$ such that

$$m = \sum_{i=0}^{n-1} V^i[a_i]e.$$

where $[a_i]$ denotes the multiplicative lift to $W(R)$.

Lemma 1.2 ([KM], Lemma 4.2.5)**.** Let $f : M' \to M$ be a map of n-cosmooth $\mathbb{E}_R$-modules such that the induced morphism $\overline{f} : M'/VM' \to M/VM$ is an isomorphism. Then f is an isomorphism.

Notation: given $\underline{r} = (r_0, \ldots, r_{n-1}) \in R^n$, we let $M_{\underline{r}}$ denote the cyclic n-cosmooth rank 1 module

$$M_{\underline{r}} := \frac{\mathbb{E}_{n,R}\langle e\rangle}{\mathbb{E}_{n,R}(Fe - \sum V^i[r_i]e)}.$$

By [KM, Proposition 4.3.1], $M_{\underline{r}}$ is n-cosmooth, and any n-cosmooth $\mathbb{E}_R$-module is Zariski-locally of this form. Throughout, we will use e to denote the generator of such n-cosmooth modules.

Combining the above lemmas we get

Corollary 1.3. Let $\underline{a}, \underline{b} \in R^n$. Then to give an isomorphism $M_{\underline{a}} \to M_{\underline{b}}$ is equivalent to giving an element $x \in M_{\underline{b}}$ such that

1. x satisfies the equation
$$(F - \sum V^i[a_i]) \cdot x = 0 \text{ in } M_{\underline{b}}$$
2. When we write $x = \sum V^i[u_i]e$ we have $u_0 \in R^\times$.

Proof. The first condition says that x defines an $\mathbb{E}_n$-module homomorphism, and the second says that this map is an isomorphism by Lemma 1.2. $\square$

1.2 Isomorphisms of n-cosmooth modules

Recall that any element of $\mathbb{E}_{n,R}$ has a unique representation in the form $\sum_{r,s} V^r[x_{r,s}]F^s$ with $x_{r,s} \in R$. Let $U_n(R) \subset \mathbb{E}_{n,R}$ be the subset of elements of the form $\sum V^i[u_i]$ with $u_0 \in R^\times$. Since V is nilpotent in $\mathbb{E}_{n,R}$ we have $U_n(R) \subset \mathbb{E}_{n,R}^\times$, but for $n > 2$, $U_n(R)$ is not closed under multiplication. By Corollary 1.3, any isomorphism $M_{\underline{a}} \to M_{\underline{b}}$ of cosmooth $\mathbb{E}_R$-modules necessarily takes the form $e \mapsto u \cdot e$ for some unique $u \in U_n(R)$. The following proposition says that $\underline{a}$ is in fact uniquely determined by u and $\underline{b}$.

Proposition 1.4. Given $u \in U_n(R)$ and $\underline{b} \in R^n$, there is a unique $\underline{a} \in R^n$ such that there is an isomorphism $M_{\underline{a}} \to M_{\underline{b}}$ given on the generator by $e \mapsto u \cdot e$.

Before proving the proposition, we establish the following computational lemma.

Lemma 1.5. Let $\underline{b} \in R^n$, and $u = \sum V^i[u_i], a = \sum V^i[a_i] \in \mathbb{E}_{n,R}$. Then in $M_{\underline{b}}$ we have

$$a \cdot (u \cdot e) = \sum V^i[c_i]e$$

with

$$c_i = a_i u_0 + P_i(\underline{b}_{\leq i}, \underline{u}_{\leq i}, \underline{a}_{<i}),$$

where $P_i(\underline{b}_{\leq i}, \underline{u}_{\leq i}, \underline{a}_{<i})$ is a polynomial in $b_0, \ldots, b_i, u_0, \ldots, u_i, a_0, \ldots, a_{i-1}$ with $\mathbb{F}_p$-coefficients.

Proof. We proceed by induction on n. The case $n = 1$ is clear as $a \cdot u = [a_0 u_0]$ and $P_0 = 0$.

Suppose $\underline{b} \in R^n$, and write $\underline{b'} := (b_0, \ldots, b_{n-2}) \in R^{n-1}$. Then we have $M_{\underline{b}}/V^{n-1}M_{\underline{b}} = M_{\underline{b'}}$, so by reduction modulo V^{n-1} and the induction hypothesis, c_i has the desired form for $i < n-1$.

It remains to compute c_{n-1}, the coefficient of V^{n-1} in $a \cdot u \in M_{\underline{b}}$. First, note that collecting terms of degree V^{n-1} in the product $(\sum V^i[a_i])(\sum V^i[u_i]e)$ we obtain a contribution of

$$V^{n-1}\left([a_0^{p^{n-1}} u_{n-1}] + [a_1^{p^{n-2}} u_{n-2}] + \cdots + [a_{n-1}u_0]\right) = V^{n-1}([a_0^{p^{n-1}} u_{n-1} + \cdots + a_{n-1}u_0]).$$

In addition, we also obtain contributions from higher Witt coordinates of terms of degree smaller than V^{n-1}. However, these terms are $\mathbb{F}_p$-polynomials in $\underline{b}, \underline{u}$, and a_i for $i < n-1$, and the lemma follows. ☐

Proof of Proposition 1.4. By Corollary 1.3, $e \mapsto u \cdot e$ defines an isomorphism $M_{\underline{a}} \to M_{\underline{b}}$ if and only if we have

$$\left(F - \sum V^i[a_i]\right) \cdot (u \cdot e) = 0 \text{ in } M_{\underline{b}}.$$

Equivalently, we want to show that the equation

$$\left(\sum V^i[a_i]\right) \cdot (u \cdot e) = F \cdot (u \cdot e)$$

in $M_{\underline{b}}$ has a unique solution $(a_0, \ldots, a_{n-1}) \in R^n$. Since u_0 is invertible, we may use Lemma 1.5 to recursively solve uniquely for a_i satisfying this equation. ☐

We let U_n denote the functor $R \mapsto U_n(R)$; U_n is represented by the scheme $(\mathbb{A}^1_{\mathbb{F}_p} \setminus 0) \times \mathbb{A}^{n-1}_{\mathbb{F}_p}$. The proposition defines a map $\phi_n(R) : U_n(R) \times R^n \to R^n$ via $(u, \underline{b}) \mapsto \underline{a}$, but this map is defined using the polynomials P_i of Lemma 1.5 so we obtain a map of $\mathbb{F}_p$-schemes $\phi_n : U_n \times \mathbb{A}^n_{\mathbb{F}_p} \to \mathbb{A}^n_{\mathbb{F}_p}$.

Remark 1.6. As the morphisms $\phi_n : U_n \times \mathbb{A}^n \to \mathbb{A}^n$ were constructed inductively in n, they are compatible in the following sense. If $g : U_n \to U_{n-1}$ is the homomorphism induced by the quotient map $\mathbb{E}_n \to \mathbb{E}_{n-1}$ and $p : \mathbb{A}^n \to \mathbb{A}^{n-1}$ is projection onto the first $n-1$ factors, then we have a commutative square

$$\begin{tikzcd}
U_n \times \mathbb{A}^n \arrow[r, "\phi_n"] \arrow[d, "g \times p"] & \mathbb{A}^n \arrow[d, "p"] \\
U_{n-1} \times \mathbb{A}^{n-1} \arrow[r, "\phi_{n-1}"] & \mathbb{A}^{n-1}.
\end{tikzcd}$$

Remark 1.7. As a scheme we have that $U_n \cong \underline{\mathrm{Aut}}(\alpha_{p^n})$, and consequently there is a natural group structure on U_n. For $n = 1, 2$, the morphism $\phi_n : U_n \times \mathbb{A}^n \to U_n$ is a group action with respect to this group structure, but this is no longer true for $n > 2$. We will discuss the cases of $n = 1$ and $n = 2$ in more detail in Section 3.

2 A presentation of Sm_n^1

Consider the map $f : \mathbb{A}^n_{\mathbb{F}_p} \to \mathrm{Sm}_n^1$, defined by

$$\underline{r} = (r_0, \ldots, r_{n-1}) \mapsto M_{\underline{r}} := \frac{\mathbb{E}_n}{(Fe = \sum V^i[r_i]e)}.$$

f is surjective since over a field, any n-cosmooth module of rank 1 has this form. Moreover, Proposition 1.4 allows us to explicitly identify the groupoid induced by f.

Lemma 2.1. We have

$$\mathbb{A}^n \times_{\mathrm{Sm}_n^1} \mathbb{A}^n \cong U_n \times \mathbb{A}^n.$$

Proof. An R-point of $\mathbb{A}^n \times_{\mathrm{Sm}_n^1} \mathbb{A}^n$ consists of $\underline{a}, \underline{b} \in R^n$ and an isomorphism $\gamma : M_{\underline{a}} \to M_{\underline{b}}$ of n-cosmooth modules over R. Writing $\gamma(e) = u \cdot e$ for $u = \sum V^i[u_i] \in U_n(R)$, we may define a morphism $\mathbb{A}^n \times_{\mathrm{Sm}_n^1} \mathbb{A}^n \to U_n \times \mathbb{A}^n$ via

$$(\underline{a}, \underline{b}, \gamma) \mapsto (u, \underline{b}).$$

Using Proposition 1.4, we see that this is an isomorphism with inverse given by

$$(u, \underline{b}) \mapsto (\phi_n(u, \underline{b}), \underline{b}, \gamma : e \mapsto u \cdot e).$$

□

It follows that even though the morphism $\phi_n : U_n \times \mathbb{A}^n \to \mathbb{A}^n$ is not a group action with respect to the natural group structure on U_n, it still induces a groupoid in schemes with objects $\mathbb{A}^n$ and morphisms $U_n \times \mathbb{A}^n$. This makes sense as we may compose isomorphisms between cosmooth modules.

Proposition 2.2. The map f induces an isomorphism

$$[\mathbb{A}^n / (U_n \times \mathbb{A}^n)] \xrightarrow{\sim} \mathrm{Sm}_n^1.$$

Under this identification, the truncation map $\mathrm{Sm}_n^1 \to \mathrm{Sm}_{n-1}^1$ is induced by the projection $\mathbb{A}^n \to \mathbb{A}^{n-1}$ onto the first $n-1$ factors.

Proof. We have already seen that f is surjective and the associated groupoid is induced by $\phi_n : U_n \times \mathbb{A}^n \to \mathbb{A}^n$, so to obtain the presentation it suffices to show that f is (formally) smooth ([Sta, Tag 04T5]). We consider a 2-commutative diagram

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r, "x"] \arrow[d] & \mathbb{A}^n \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] & \mathrm{Sm}_n^1 \end{tikzcd}$$

where $\pi : A \to B$ is a surjection of local rings. This is the data of an n-cosmooth D_A-module M, $\underline{b} \in B^n$, and an isomorphism

$$\alpha : M_{\underline{b}} \xrightarrow{\sim} M|_B.$$

A lift $\operatorname{Spec} A \to \mathbb{A}^n$ in the diagram would be the data of $\underline{a} \in A^n$ such that $\pi(\underline{a}) = \underline{b}$ and an isomorphism

$$\gamma : M_{\underline{a}} \xrightarrow{\sim} M$$

lifting α.

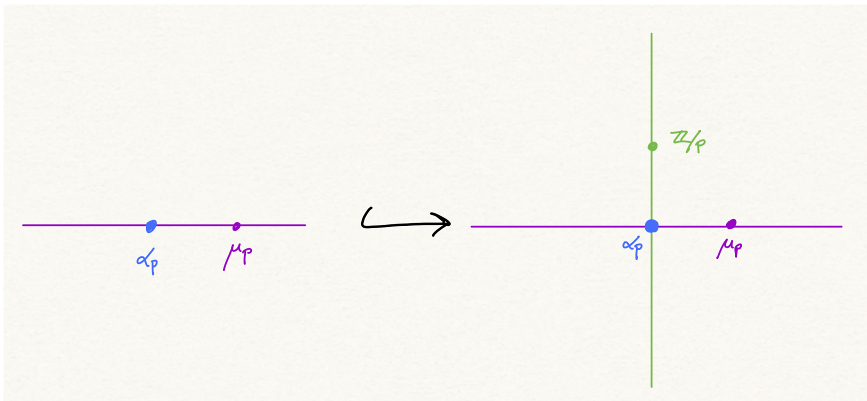
Since A is a local ring, there exists $\underline{c} \in A^n$ such that $M \cong M_{\underline{c}}$. Moreover $\pi : A \to B$ is a surjection of local rings, so it induces a surjection $U_n(A) \to U_n(B)$. Therefore if $\overline{\lambda} \in U_n(B)$ defines the isomorphism α under Proposition 1.4, we lift to $\lambda \in U_n(A)$ and let $\underline{a} := \phi_n(\lambda, \underline{c})$. Then $\pi(\underline{a}) = \phi_n(\overline{\lambda}, \pi(\underline{c})) = \underline{b}$, and λ yields an isomorphism $\gamma : M_{\underline{a}} \to M_{\underline{c}} \cong M$ lifting α.

The description of the truncation maps then follows from Remark 1.6. □

3 Examples

3.1 1-smooth group schemes of order p

For $n = 1$, we have $U_1 \cong \mathbb{A}^1 \setminus 0$ and $\phi_1 : U_1 \times \mathbb{A}^1 \to \mathbb{A}^1$ is $(\lambda, x) \mapsto \lambda^{p-1}x$. Thus Proposition 2.2 tells us that the moduli space of 1-smooth group schemes of order p is $[\mathbb{A}^1/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts through weight $p-1$. This is consistent with the Oort-Tate presentation of the moduli stack of group schemes of order p [OT], which identifies the moduli of group schemes of order p with with the space of line bundles $\mathcal{L}$ with sections of $\mathcal{L}^{\otimes(1-p)}, \mathcal{L}^{\otimes(p-1)}$ which multiply to zero.



The inclusion of Sm_1^1 into the Oort-Tate moduli space.

3.2 2-smooth group schemes of order p^2

When $n = 2$, $U_2(R) \subset \mathbb{E}_{2,R}^{\times}$ is a multiplicative subgroup, owing to the identity $V([x] + [y]) = V([x+y])$ in $\mathbb{E}_2$.[1] It follows that the map $\phi_2 : U_2 \times \mathbb{A}^2 \to \mathbb{A}^2$ is a group action and $\mathrm{Sm}_2^1 = [\mathbb{A}^2/U_2]$ is a quotient stack.

Remark 3.1. The identifications $U_2 = \underline{\mathrm{Aut}}(\alpha_{p^2})$ and $T_{\alpha_{p^2}} \mathrm{Sm}_2^1 \cong \mathbb{A}^2$ also provide a group action $U_2 \times \mathbb{A}^2 \to \mathbb{A}^2$ but this is not the same as the action ϕ_2. Explicitly, ϕ_2 is given by

$$(r, s) \cdot (c, d) = (r^{p-1}c, r^{-1}(r^{p^2}d + s^p c - r^{p^2-p}sc^p))$$

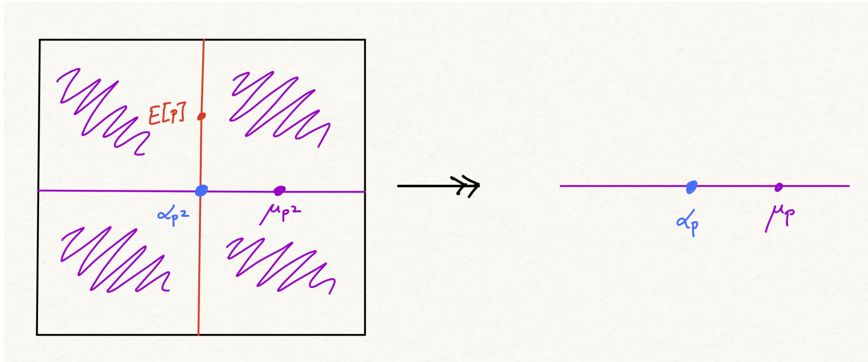
for $(r, s) \in U_2, (c, d) \in \mathbb{A}^2$, while the action on the tangent space is

$$(r, s) \cdot (c, d) = (r^{p-1}c, r^{-1}(r^{p^2}d + s^p c)).$$

Note that the second map is just the differential of the first map, as we throw out the higher order term $r^{p^2-p}sc^p$.

Over an algebraically closed field of characteristic p, the only 2-smooth group schemes of order p^2 are μ_{p^2}, α_{p^2}, and $E[p]$, the p-torsion subgroup scheme in a supersingular elliptic curve. Moreover, these group schemes are distinguished by looking at the Verschiebung operator, which is an isomorphism for μ_{p^2}, zero for α_{p^2}, and nilpotent of order 2 for $E[p]$. This gives the following picture of the moduli space $\mathrm{Sm}_2^1 \cong [\mathbb{A}^2/U_2]$ and the truncation morphism $\mathrm{Sm}_2^1 \to \mathrm{Sm}_1^1$:

[1]This group structure coincides with that coming from the identification $U_2 \cong \underline{\mathrm{Aut}}(\alpha_{p^2})$.



The truncation morphism $\mathrm{Sm}_2^1 \to \mathrm{Sm}_1^1$.

Acknowledgments

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References

[KM] C. Kothari and J. Mundinger. Dieudonné theory for n-smooth group schemes. ArXiv preprint, version 1. https://www.arxiv.org/abs/2408.15333.

[OT] F. Oort and J. Tate. Group schemes of prime order. Ann. Sci. Éc. Norm. Supér., Ser. 4, **3**, 1-21 (1970).

[Sta] Stacks Project Authors. The Stacks Project. https://stacks.math.columbia.edu/.